Elementary equivalence in positive logic via prime products

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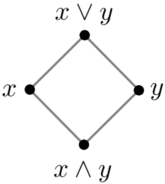
Lattices

A *lattice* is a partially ordered set (*poset*) in which any two elements x and y have

- a least upper bound, denoted $x \vee y$
- a greatest lower bound, denoted $x \wedge y$.



It is *distributive* if it satisfies

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),$$

bounded if it has a maximum $\top$ and minimum $\bot$ element.

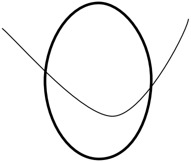
A *Boolean algebra* is a bounded distributive lattice in which every element x has a *complement* $\neg x$, i.e., an element for which

$$x \wedge \neg x = \bot \text{ and } x \vee \neg x = \top.$$

Lattice filters

Let $\mathbf{A}$ be a lattice.

A non-empty subset F of A is a *filter* of $\mathbf{A}$, when



- if $x \in F$ and $x \leqslant y$ then $y \in F$, and
- if $x, y \in F$ then $x \wedge y \in F$.

It is *proper* if $F \neq A$. A proper filter F is said to be *prime* when

- if $x \vee y \in F$ then $x \in F$ or $y \in F$.

A filter is an *ultrafilter* if it is maximal among proper filters.

If $\mathbf{A}$ is a distributive lattice, then its ultrafilters are prime filters.

The converse is true, when $\mathbf{A}$ is a Boolean algebra, in particular

$$x \in F \text{ or } \neg x \in F, \text{ for all } x \in A.$$

Recall that for any set I, its powerset $\mathcal{P}(I)$, when ordered by $\subseteq$, is a Boolean algebra where the complement of $X \subseteq I$ is $I \setminus X$.

Ultraproducts as direct limits

Given a family of structures $\{M_i : i \in I\}$, and ultrafilter $\mathcal{U}$ of $\mathcal{P}(I)$.

- The poset $\langle \mathcal{U}, \supseteq \rangle$ is *directed* as $\mathcal{U}$ is closed under intersections.
- We define a *directed system* indexed by $\langle \mathcal{U}, \supseteq \rangle$ with elements

$$\prod_{i \in X} M_i \text{ for every } X \in \mathcal{U},$$

and maps, whenever $Y \supseteq X$ in $\mathcal{U}$, the canonical projection

$$f_{Y,X} : \prod_{i \in Y} M_i \to \prod_{i \in X} M_i.$$

- The ultraproduct $\prod_{i \in I} M_i / \mathcal{U}$ is the direct limit of the system.

Why? Pick $a, b \in \prod_{i \in I} M_i / \mathcal{U}$,

$$\begin{aligned} a = b \text{ in the limit} &\iff a{\restriction}_V = b{\restriction}_V \text{ for some } V \in U \\ &\iff V \subseteq \{i \in I : a(i) = b(i)\} = \\ &\qquad [\![a = b]\!] \text{ for some } V \in \mathcal{U} \\ &\iff [\![a = b]\!] \in \mathcal{U}. \end{aligned}$$

Classic Results

For any first order formula $\varphi(x_1, \dots, x_n)$ and $\mathbf{a}_1, \dots, \mathbf{a}_n \in \prod_{i \in I} M_i$,

$$[\![\varphi(\mathbf{a}_1, \dots, \mathbf{a}_n)]\!] = \{i \in I : M_i \models \varphi(\mathbf{a}_1(i), \dots, \mathbf{a}_n(i))\}.$$

Łoś' Theorem

$$\prod_{i \in I} M_i/\mathcal{U} \models \varphi(\mathbf{a}_1/\mathcal{U}, \dots, \mathbf{a}_n/\mathcal{U}) \text{ iff } [\![\varphi(\mathbf{a}_1, \dots, \mathbf{a}_n)]\!] \in \mathcal{U}.$$

If every $M_i = M$ for some structure M, then $\prod_{i \in I} M/\mathcal{U}$ is called an *ultrapower* of M, and M is an *ultraroot* of $\prod_{i \in I} M/\mathcal{U}$.

Thm

Let K be a class of structures. Then K is axiomatized by first order sentences (i.e., *elementary*) iff K is closed under ultraroots and ultraproducts.

Keisler–Shelah Theorem

Two structures satisfy the same first order sentences iff they have isomorphic ultrapowers.

Let $\mathbb{I} = \langle I; \leqslant \rangle$ be a poset and let

$$\mathrm{Up}(\mathbb{I}) = \{\text{upwardly closed subsets of } I\}.$$

Then $\mathrm{Up}(\mathbb{I})$ is a distributive lattice, ordered by inclusion.

Note that if $\mathbb{I}$ is an anti-chain, then $\mathrm{Up}(\mathbb{I}) = \mathcal{P}(I)$, in which case, any prime filter of $\mathrm{Up}(\mathbb{I})$, would be an ultrafilter over I.

A *well-ordered forest* is a poset $\mathbb{I} = \langle I; \leqslant \rangle$ such that the downward closure of every element of I is well-ordered.

An *ordered system* is a family of structures $\{M_i : i \in \mathbb{I}\}$, indexed by a well-ordered forest $\mathbb{I}$ such that whenever $i \leqslant j$ there is a homomorphism $h_{ij} : M_i \to M_j$, and

$$h_{ii} = id_{M_i} \text{ and } i \leqslant j \leqslant k \text{ implies } h_{jk} \circ h_{ij} = h_{ik}.$$

Prime products

Given an *ordered system* $\{M_i : i \in \mathbb{I}\}$ and a prime filter $\mathcal{F}$ of $\mathrm{Up}(\mathbb{I})$.

We construct a directed system indexed by $\langle \mathcal{F}, \supseteq \rangle$, as before with canonical projections as maps on the objects:

For each $V \in \mathcal{F}$ let S_V be the substructure of $\prod_{i \in I} M_i$ on

$$\{\mathbf{a} \in \prod_{i \in V} M_i : h_{ij}(\mathbf{a}(i)) = \mathbf{a}(j) \text{ whenever } i \leqslant j \text{ in } V\}.$$

We define the *prime product* $\prod_{i \in \mathbb{I}} M_i / \mathcal{F}$ to be the direct limit of this system.

Note when the poset $\mathbb{I}$ is discreetly ordered, prime products coincide with ultraproducts.

Prime products

We can consider the universe of $\prod_{i\in\mathbb{I}} M_i/\mathcal{F}$ to be $S_{\mathcal{F}} = \bigcup_{V\in\mathcal{F}} S_V$.
For each $a \in S_{\mathcal{F}}$

- we let V_a be the unique $V \in \mathcal{F}$ such that $a \in S_V$, and
- $a/\mathcal{F}$ the incarnation of a in the direct limit $\prod_{i\in\mathbb{I}} M_i/\mathcal{F}$.

For every formula $\varphi(x_1, \dots, x_n)$ and $a_1, \dots a_n \in S_{\mathcal{F}}$ let

$$[\![\varphi(a_1, \dots, a_n)]\!] = \{i \in V_{a_1} \cap \cdots \cap V_{a_n} : M_i \models \varphi(a_1(i), \dots, a_n(i))\}.$$

Positive logic

Positive logic is concerned with formulas that are preserved by *homomorphisms*, i.e., built from atomic formulas using only

$$\exists, \wedge, \vee, \text{ and } \perp.$$

We call these *positive (existential)* formulas.

Positive Łoś' Theorem

For every *positive* formula $\varphi(x_1, \ldots, x_n)$ and $a_1, \ldots a_n \in S_{\mathcal{F}}$,

$$\prod_{i \in \mathbb{I}} M_i/\mathcal{F} \models \varphi(a_1/\mathcal{F}, \ldots, a_n/\mathcal{F}) \text{ iff } [\![\varphi(a_1, \ldots, a_n)]\!] \in \mathcal{F}.$$

Note, the assumption that $\mathbb{I}$ is a well-ordered forest is crucial for the preservation of the existential quantifier $\exists$.

Prime products also preserve implications between positive formulas, i.e., *h-inductive* formulas that have the form

$$\forall \vec{x}(\varphi(\vec{x}) \to \psi(\vec{x})),$$

where φ and ψ are positive formulas.

Corollary

A class of structures can be axiomatized by h-inductive sentences iff it is closed under prime products and ultraroots.
A prime product $\prod_F M_i / \equiv_F$ is called a prime *power* of M when every $M_i = M$.

Positive Keisler-Shelah

Thm

- Under the GCH, two structures satisfy the same positive sentences iff they have isomorphic prime powers of ultrapowers.
- Without the GCH we can show: Two structures satisfy the same positive sentences iff they have isomorphic prime *products* of ultrapowers.

We can not remove the ultrapowers above. Consider $\mathbb{Q}$ in the language $\mathbb{Q} \cup \{\leq\}$, and $\mathbb{Q}^* = \mathbb{Q} \cup \{\infty\}$. Then $\mathbb{Q}$ and $\mathbb{Q}^*$ have the same positive theory, but one can show that prime powers preserve and reflect the existence of maxima.

Application

Thm (M. Raftery W., 2020, MLQ, Thm. 7.6)

Let K be a quasivariety of finite type, with a finite nontrivial member. Then the following conditions are equivalent.

1) K is Passively Structurally Complete (PSC), i.e., every quasi-equation whose premises are not unifiable over K is valid in K.
2) The nontrivial members of K satisfy the same existential positive sentences.
3) The nontrivial members of K have a common retract.

Thm

A quasivariety K is PSC iff any two nontrivial members of K have isomorphic prime products of ultrapowers.

thank you