Embedding Kozen-Tiuryn Logic into Residuated One-sorted Kleene Algebra With Tests[1]

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Kleene algebra

Definition

A Kleene algebra [Koz94] is a structure $\mathscr{K} = (K, \vee, \cdot, {}^*, 1, 0)$ such that

- $(K, \vee, \cdot, 1, 0)$ is an *idempotent semiring*, i.e.,
  - $(K, \cdot, 1)$ is a monoid,
  - $(K, \vee, 0)$ is an idempotent commutative monoid (hence, a join-semilattice),
  - $x(y \vee z) = xy \vee xz$, $(y \vee z)x = yx \vee zx$, and
  - $x0 = 0 = 0x$, and
- ${}^*: K \to K$ such that

$$1 \vee a \vee a^*a^* \leq a^* \qquad ax \leq x \Rightarrow a^*x \leq x \qquad xa \leq x \Rightarrow xa^* \leq x$$

$\mathscr{K}$ is *-continuous iff $ab^*c = \bigvee_{n \geq 0} ab^nc$.

Example

The relational Kleene algebra over a set X is $\mathscr{R}(X) = (2^{X \times X}, \cup, \circ, {}^*, \mathrm{id}, \emptyset)$;

- $\circ$ denotes composition, and
- $R^* = \bigcup_{i \geq 0} R^i$, where $R^0 = \mathrm{id}$ and $R^{i+1} = R \circ R^i$.

Kleene algebra with tests

Definition

A Kleene algebra with tests [Koz97] is $\mathscr{B} = (K, B, \vee, \cdot, ^*, 1, 0, ^-)$ where

- $(K, \vee, \cdot, ^*, 1, 0)$ is a Kleene algebra
- $B \subseteq K$
- $(B, \vee, \cdot, ^-, 1, 0)$ is a Boolean algebra.

Prop. Every KA is a KAT, where the test subalgebra is $B = \{0, 1\}$.

Example

The relational KAT over a set X is $\mathscr{R}(X)$ together with the Boolean test subalgebra 2^{id}.

Prop. [KS97] The equational theory of KAT is identical to the equational theory of rKAT.

Propositional while programs

$$\text{Tests} \quad \beta := \mathrm{b} \mid \bar{\beta} \mid \beta \wedge \beta \mid \beta \vee \beta$$

$$\text{Programs} \quad \pi := \textbf{skip} \mid \mathrm{p} \mid \pi;\pi \mid \textbf{if } \beta \textbf{ then } \pi \textbf{ else } \pi \mid \textbf{while } \beta \textbf{ do } \pi$$

In KAT:

$$b \wedge c := bc \qquad b \vee c := b \vee c$$

$$p;q := pq$$

$$\textbf{skip} := b \vee \bar{b}$$

$$\textbf{if } b \textbf{ then } p \textbf{ else } q := (bp) \vee (\bar{b}q)$$

$$\textbf{while } b \textbf{ do } p := (bp)^*\bar{b}$$

Non-termination: $p = 0$

Partial correctness: $\{b\}p\{c\} \iff bp = bpc$

Kozen-Tiuryn Logic

In [KT03] Kozen and Tiuryn introduces a logic (see next 2 slides) which represents partial correctness by a formula, instead of an equation in KAT. They argue that this has certain advantages, e.g., it facilitates a better distinction between local and global properties.

Kozen-Tiuryn Logic

Definition

Let $\mathsf{B} = \{\mathsf{b}_i \mid i \in \omega\}$ be the set of test variables and let $\mathsf{P} = \{\mathsf{p}_i \mid i \in \omega\}$ be the set of program variables.
We define the following sorts of syntactic objects:

tests	$b, c := \mathsf{b}_i \mid 0 \mid b \Rightarrow c$
programs	$p, q := \mathsf{p}_i \mid b \mid p \oplus q \mid p \otimes q \mid p^+$
formulas	$f, g := b \mid p \Rightarrow f$
environments	$\Gamma, \Delta := \epsilon \mid \Gamma, p \mid \Gamma, f$
sequents	$\Gamma \vdash f$

We define $1 := 0 \Rightarrow 0$, $\neg b := b \Rightarrow 0$ and $p^* := 1 \oplus p^+$.
The logic S is defined in [KT03] to be the set of sequents provable in the following proof system:

(Id) $b \vdash b$

(TC) $$\frac{\Gamma, b, \Delta \vdash f \qquad \Gamma, \overline{b}, \Delta \vdash f}{\Gamma, \Delta \vdash f}$$

(R⇒) $$\frac{\Gamma, p \vdash f}{\Gamma \vdash p \Rightarrow f}$$

(I⊗) $$\frac{\Gamma, p, q, \Delta \vdash f}{\Gamma, p \otimes q, \Delta \vdash f}$$

(I⊕) $$\frac{\Gamma, p, \Delta \vdash f \qquad \Gamma, q, \Delta \vdash f}{\Gamma, p \oplus q, \Delta \vdash f}$$

(I$^+$) $$\frac{g, p \vdash f \qquad g, p \vdash g}{g, p^+ \vdash f}$$

(E$^+$) $$\frac{\Gamma, p^+, \Delta \vdash f}{\Gamma, p, \Delta \vdash f}$$

(CC$^+$) $$\frac{\Gamma, p^+, \Delta \vdash f}{\Gamma, p^+, p^+, \Delta \vdash f}$$

(I0) $\Gamma, 0, \Delta \vdash f$

(cut) $$\frac{\Gamma \vdash g \qquad \Gamma, g, \Delta \vdash f}{\Gamma, \Delta \vdash f}$$

(I⇒) $$\frac{\Gamma, p, f, \Delta \vdash g}{\Gamma, p \Rightarrow f, p, \Delta \vdash g}$$

(E⊗) $$\frac{\Gamma, p \otimes q, \Delta \vdash f}{\Gamma, p, q, \Delta \vdash f}$$

(E$\oplus_1$) $$\frac{\Gamma, p \oplus q, \Delta \vdash f}{\Gamma, p, \Delta \vdash f}$$

(E$\oplus_2$) $$\frac{\Gamma, p \oplus q, \Delta \vdash f}{\Gamma, q, \Delta \vdash f}$$

(Wf) $$\frac{\Gamma, \Delta \vdash g}{\Gamma, f, \Delta \vdash g}$$

(Wp) $$\frac{\Gamma \vdash f}{p, \Gamma \vdash f}$$

Binary relation semantics

Definition

A Kozen–Tiuryn model is a pair $M = (W, V)$, where W is a non-empty set and $V : \mathsf{B} \cup \mathsf{P} \to \mathscr{R}(W)$ such that $V(\mathsf{b}) \subseteq \mathrm{id}_W$ for all $\mathsf{b} \in \mathsf{B}$.
We define the M-interpretation function $[\,]_M : Ex_{\mathsf{S}} \to \mathscr{R}(W)$ as follows:

- $[\mathsf{b}]_M = V(\mathsf{b})$, $\quad [\mathsf{p}]_M = V(\mathsf{p})$, $\quad [0]_M = \emptyset$
- $[b \Rightarrow c]_M = \{(s,s) \mid (s,s) \notin [b]_M$ or $(s,s) \in [c]_M\}$
- $[p \oplus q]_M = [p]_M \cup [q]_M$
- $[p \otimes q]_M = [p]_M \circ [q]_M$
- $[p^+]_M = [p]_M^+$
- $[p \Rightarrow f]_M = \{(s,s) \mid \forall t.(s,t) \in [p]_M \implies (t,t) \in [f]_M\}$
- $[\epsilon]_M = \mathrm{id}_W$
- $[\Gamma, \Delta]_M = [\Gamma]_M \circ [\Delta]_M$

(Here $^+$ denotes transitive closure and $\circ$ denotes relational composition.)

Definition

A sequent $\Gamma \vdash f$ is valid in M iff,
for all $s, t \in W$, if $(s,t) \in [\![\Gamma]\!]_M$, then $(t,t) \in [\![f]\!]_M$.

Theorem 1

$\Gamma \vdash f$ is provable in S iff $\Gamma \vdash f$ is valid in every Kozen–Tiuryn model.

- Observe that $[\![f]\!]_M \subseteq \mathrm{id}_W$ for all formulas f
- If $(s,s) \in [\![f]\!]_M$, then we may say that formula f is true in s.
- Note that $[\![bp \Rightarrow c]\!]_M$ is the set of (s,s) such that, for all t, if $(s,s) \in [\![b]\!]_M$ and $(s,t) \in [\![p]\!]_M$, then $(t,t) \in [\![c]\!]_M$.
- Hence, $bp \Rightarrow c$ represents a partial correctness assertion: the formula is true in s iff b is true in s and p connects s with a state t only if c is true in t.

Questions

- How does S relate to mainstream substructural logic?
- Is there a one-sorted residuated structure into which we can interpret S?

Residuals

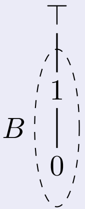
Definition

An idempotent semiring is residuated if it has two binary operation $\to$, $\hookrightarrow$ such that

$$xy \leq z \iff x \leq y \to z \iff y \leq x \hookrightarrow z. \qquad (1)$$

Note. One can not translate $p \Rightarrow b$ to $p \to b$, since $0 \to 0$ is maximal.

Example



$\to$	0	1	$\top$
0	$\top$	$\top$	$\top$
1	0	1	$\top$
$\top$	0	0	$\top$

$\cdot$	0	1	$\top$
0	0	0	0
1	0	1	$\top$
$\top$	0	$\top$	$\top$

$*$	0	1	$\top$
	1	1	$\top$

Let $R \subseteq S \times S$ and $B \subseteq \mathrm{id}_S$.

$$R \to B = \{(s,u) \mid \forall v.\,(u,v) \in R \implies (s,v) \in B\}$$
$$R \Rightarrow B = \{(u,u) \mid \forall v.\,(u,v) \in R \implies (v,v) \in B\}$$

Now consider the operations $c, e : 2^{S\times S} \to 2^{S\times S}$

$$c(R) = \{(u,u) \mid \exists s.\,(s,u) \in R\} \tag{2}$$
$$e(R) = \{(s,u) \mid (u,u) \in R\} \tag{3}$$

Note that c and e form a *Galois connection* in the sense that,

$$c(R) \subseteq Q \iff R \subseteq e(Q)\,. \tag{4}$$

Proposition 1

$$c(R \to e(B)) = R \Rightarrow B \tag{5}$$

Kleene algebra with codomain

The following one-sorted alternative to KAT is presented in [DS11].

Definition

A Kleene algebra with (anti)codomain is $\mathscr{A} = (K, \vee, \cdot, ^*, 1, 0, a)$ where $a : K \to K$ such that

$$xa(x) = 0$$
$$a(xy) = a(a^2(x)y)$$
$$a^2(x) \vee a(x) = 1$$

A codomain operation is then defined by $c(x) := a^2(x)$.

Prop. $(c(K), \vee, \cdot, 1, 0)$ is a Boolean algebra where $a(x)$ is the complement of $x \in c(K)$.

Thm. The equational theory of KAT embeds into that of KAC.

Relational KAC

Example

Extend a relational Kleene algebra with

$$a(R) = \{(s, s) \mid \neg \exists t.(t, s) \in R\}.$$

Thus creating a relational KAC.

The codomain operation $a(a(R)) = c(R)$, is then as expected

$$c(R) = \{(s, s) \mid \exists t.(t, s) \in R\}.$$

Residuated program algebras

Definition

An SKAT $\mathscr{P} = (K, \vee, \cdot, \to, \hookrightarrow {}^*, a, e, 1, 0)$ comprises

- a residuated Kleene algebra $(K, \vee, \cdot, \hookrightarrow, \to, {}^*, 1, 0)$,
- a Kleene algebra with codomain $(K, \vee, \cdot, {}^*, a, 1, 0)$, and
- a unary operation e on K that satisfies the following:

$$a(a(e(x))) \leq x \tag{6}$$

$$x \leq e(a(a(x))) \tag{7}$$

$$e(x) \leq e(x \vee y) \tag{8}$$

We define $c(x) := a(a(x))$. An SKAT is *-continuous (denoted SKAT*) iff its underlying Kleene algebra is *-continuous.

Prop.: The class of all SKATs is a variety.

Embedding result

Let Tm be the absolutely free SKAT-type algebra over $\{x_1, x_2, \dots\}$.

Definition

We define $Tr : Ex_{\mathsf{S}} \to Tm$ as follows:

- $Tr(\mathsf{p}_n) = \mathsf{x}_{2n}, \quad Tr(\mathsf{b}_n) = c(\mathsf{x}_{2n+1}), \quad Tr(0) = c(0), \quad Tr(\epsilon) = 1$
- $Tr(b \Rightarrow c) = c(Tr(b) \to e(Tr(c)))$
- $Tr(p \oplus q) = Tr(p) \vee Tr(q)$
- $Tr(p \otimes q) = Tr(p) \cdot Tr(q)$
- $Tr(p^+) = Tr(p) \cdot Tr(p)^*$
- $Tr(p \Rightarrow f) = c(Tr(p) \to e(Tr(f)))$
- $Tr(\Gamma, \Delta) = Tr(\Gamma) \cdot Tr(\Delta)$

Theorem 2

A sequent $\Gamma \vdash f$ is provable in S iff $c\big(Tr(\Gamma)\big) \leq Tr(f)$ belongs to the equational theory of *-continuous SKAT.

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