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Varieties of De Morgan Monoids

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Abstract

A *De Morgan monoid* $\mathbf{A} = \langle A; \cdot, \wedge, \vee, \neg, e\rangle$ comprises a distributive lattice $\langle A; \wedge, \vee\rangle$, a commutative monoid $\langle A; \cdot, e\rangle$ satisfying $x \leqslant x^2 := x \cdot x$, and a function $\neg\colon A \longrightarrow A$, called an *involution*, such that $\mathbf{A}$ satisfies $\neg\neg x = x$ and $x \cdot y \leqslant z \iff x \cdot \neg z \leqslant \neg y$. (The derived operations $x \rightarrow y := \neg(x \cdot \neg y)$ and $f := \neg e$ turn $\mathbf{A}$ into an involutive residuated lattice in the sense of [2].) The class DMM of all De Morgan monoids is a variety that algebraizes the relevance logic $\mathbf{R^t}$ of [1]. Its lattice of subvarieties Λ_{DMM} is dually isomorphic to the lattice of axiomatic extensions of $\mathbf{R^t}$. We study the former here, as a route to the latter.

We prove that Λ_{DMM} has just four atoms, and that there are just 68 minimal quasivarieties of De Morgan monoids. Given a variety K of De Morgan monoids, it helps, in this investigation, to know whether every proper subquasivariety of K generates a proper subvariety of K. If that is true, then K is said to be *structurally complete*. In the hope of identifying all the structurally complete varieties of De Morgan monoids in the future, we fully characterize those that satisfy a weak ('passive') form of structural completeness.

The description involves four cases, two of which are sporadic. The third is the denumerable family of varieties of odd Sugihara monoids, which is very well understood. The fourth family is uncountable. In each of its members, the nontrivial algebras have, as a retract, a certain four-element totally ordered De Morgan monoid known as $\mathbf{C}_4$. All varieties falling under the first three cases are structurally complete, but the last case encompasses some counter-examples. We prove that there is a largest variety M of the fourth kind, for which we supply a short equational axiomatization. We also show that, in the lattice of subvarieties of M, the variety generated by $\mathbf{C}_4$ has exactly six finitely generated covers, all of which are structurally complete.

Keywords: De Morgan monoid; relevance logic; structural completeness.

Category: **Algebra, Mathematical Logic.**

References

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